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Comment on 'A model of the motion of a heavy gas cloud released on a uniform slope'

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Abstract

A model originally developed for the equilibrium motion of a heavy gas cloud instantaneously released on a uniform slope is applied for new terrain types, and the cloud shape and terminal velocity are predicted. This may be used as an additional test case for the evaluation of two-dimensional shallow water models.

Keywords: Analytical solution; Box model; Heavy gas; Topography

1. Introduction

Previously, Webber et al. [1] from SRD, UK, presented an integral model for heavy gas dispersion on uniform slopes. The novelty of this model (hereafter called the SRD model) was the shape of the model cloud which had a horizontal top surface, a front of a universal shape, and a rear boundary which intersected the terrain. In contrast to the continuous development of a similar release in flat terrain this wedge shape of the cloud was predicted to be steady. This was also observed to be the equilibrium state of a numerical shallow water model. A complete description of the problem is given in Webber et al. [1].

2. Motion in a valley with v-shaped cross section

In this note the SRD model will be applied in a v-shaped valley. The notation is similar to that of the original article, except for the parameter α which describes the modified topography. The new model cloud will be shaped as shown in Fig. 1. The top

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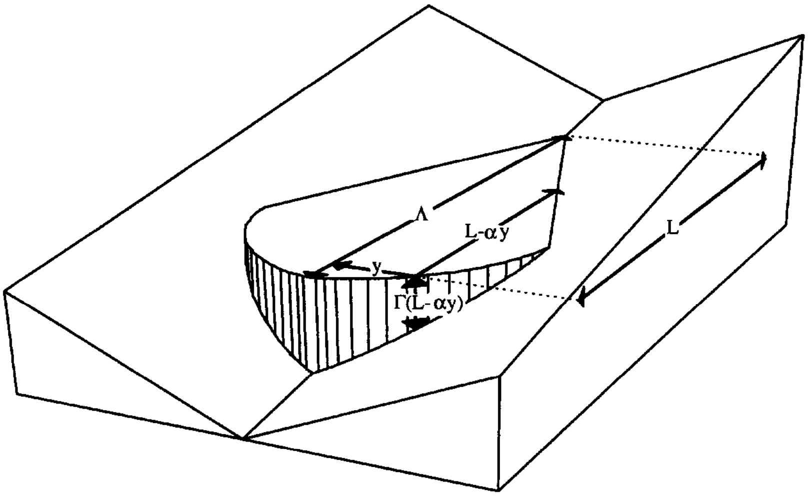

Fig. 1. Perspective of the modified heavy gas model cloud in a v-shaped valley.

surface is still horizontal, but the intersection with the terrain is now described by two lines and the shape of the front is slightly different.

As in Webber et al. [1], the height of the terrain is written

$$a(\boldsymbol{x}) = \Gamma \boldsymbol{x} \cdot \hat{\boldsymbol{n}} \tag{1}$$

but instead of the uniform slope the terrain is described by

$$\hat{\boldsymbol{n}} = \begin{cases} (-1, +\alpha) & \text{for} \quad x_2 > 0, \\ (-1, -\alpha) & \text{for} \quad x_2 < 0, \end{cases} \tag{2}$$

where x_2 is the direction across the valley. The front velocity is

$$u_f = k_f \sqrt{g'(h-a)} = k_f \sqrt{g'\Gamma(L - \alpha|y|)} \tag{3}$$

where the terrain height a has been evaluated from the coordinates of the front (L, y). When the shape of the cloud is constant, the projection of front velocity on the x_1-axis must be equal to the cloud advection velocity u:

$$\frac{u}{\sqrt{1 + (\mathrm{d}L/\mathrm{d}y)^2}} = k_f \sqrt{g'\Gamma(L - \alpha|y|)}. \tag{4}$$

This leads to

$$\frac{\mathrm{d}\hat{L}}{\mathrm{d}|\hat{y}|} = -\sqrt{\frac{1 - \hat{L} + \alpha|\hat{y}|}{\hat{L} - \alpha|\hat{y}|}} \tag{5}$$

where $\hat{L} = L/\Lambda$ and $\hat{y} = y/\Lambda$ are the front coordinates normalized by the cloud dimension in the down valley direction $\Lambda = u^2/k_f^2 g'\Gamma$. The solution is symmetrical around $x_2 = 0$, so we only need to solve the problem for $x_2 > 0$. The parameterization with ω used in Webber et al. [1] is still practical.

$$\hat{L} - \alpha\hat{y} = \cos^2\omega \;\Rightarrow\; \frac{d\hat{L}}{d\hat{y}} = -\,2\cos\omega\sin\omega\frac{d\omega}{d\hat{y}} + \alpha. \tag{6}$$

With insertion of $\hat{L} - \alpha\hat{y}$ in Eq. (5) we obtain

$$\frac{d\hat{L}}{d\hat{y}} = \frac{-\,\sin\omega}{\cos\omega}. \tag{7}$$

A differential equation for the front cloud boundary is found from a combination of Eqs. (6) and (7).

$$\frac{d\hat{y}}{d\omega} = \frac{2\cos^2\omega\sin\omega}{\sin\omega + \alpha\cos\omega} = 2\cos^2\omega - 2\alpha\frac{\cos^2\omega}{\tan\omega + \alpha}. \tag{8}$$

Since the local front velocity u_f is a continuous function in the terrain of interest, I postulate that the front of the equilibrium cloud must be smooth even at the center line, i.e. that $d\hat{L}/d\hat{y} = 0$ for $\hat{y} = 0$, and according to Eq. (7) this leads to the boundary condition $\omega = 0$ for $\hat{y} = 0$. With this boundary condition the solution to Eq. (8) becomes

$$\hat{y} = \omega + \sin\omega\cos\omega - 2\alpha\left\{\frac{(6\alpha + 2\alpha^3)\omega + 4\ln[\sin\omega + \alpha\cos\omega] - 4\ln\alpha}{4(1+\alpha^2)^2} + \frac{\alpha\sin 2\omega + \cos 2\omega - 1}{4(1+\alpha^2)}\right\} \tag{9}$$

where the first two terms are identified as the original solution and the last term is zero for $\alpha = 0$. The shape of the front $(\hat{L}, \hat{y})$ has been plotted in Fig. 2 for different values of arctan α. The curve for arctan $\alpha = 0$ is similar to the curve in Fig. 5 of Webber et al. [1]. The cloud volume is found from

$$dV = \tfrac{1}{2}\Gamma[L - \alpha y]^2\, dy \tag{10}$$

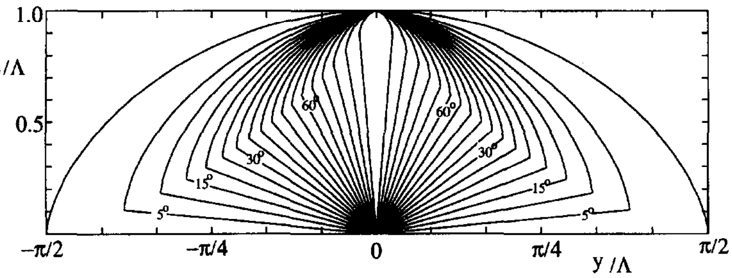

Fig. 2. Top view of the equilibrium cloud in a v-shaped valley with different values of arctan α. The straight lines in each curve are intersections with the terrain.

and since the cloud is symmetrical, we just insert the solution for $x_2 > 0$ and multiply by 2. The cloud volume is

$$V = \Gamma \Lambda^3 2 \int_0^{\pi/2} \cos^6 \omega - \alpha \frac{\cos^6 \omega}{\tan \omega + \alpha} \mathrm{d}\omega = \Gamma \Lambda^3 \Omega_6(\alpha) \tag{11}$$

where the normalized cloud volume Ω_6 now depends on α.

$$\Omega_6(\alpha) = \frac{5\pi}{16} + \frac{2\alpha \ln \alpha}{(1+\alpha^2)^4} + \frac{\alpha(11 + 7\alpha^2 + 2\alpha^4)}{6(1+\alpha^2)^3} - \frac{\alpha^2(35 + 35\alpha^2 + 21\alpha^4 + 5\alpha^6)\pi}{16(1+\alpha^2)^4}. \tag{12}$$

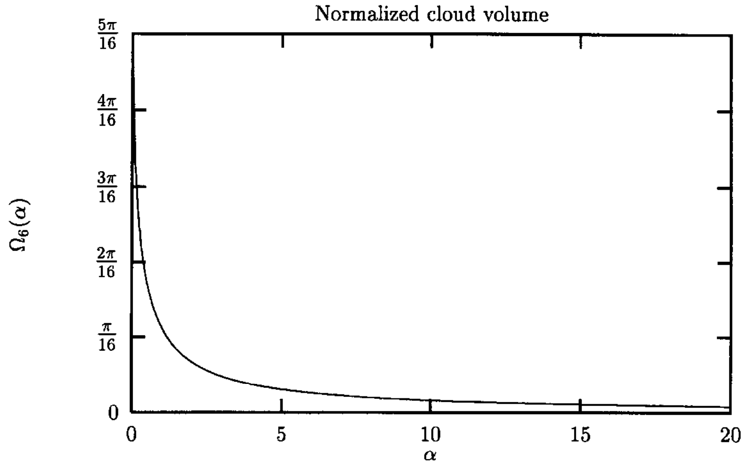

Fig. 3. The normalized cloud volume in a v-shaped valley $\Omega_6(\alpha)$ as a function of the slope ratio α.

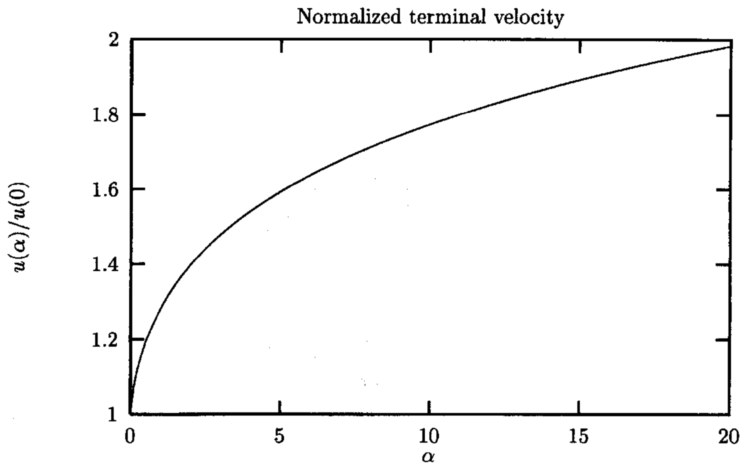

Fig. 4. The normalized terminal velocity of the cloud $u(\alpha)/u(0)$ in a v-shaped valley.

This function is shown in Fig. 3. In the limit of uniform slope $\alpha \to 0$, the function becomes the original normalized volume $\Omega_6 = \frac{5}{6}\pi$ as presented by Webber et al. [1]. It is noted that the cloud velocity

$$u(\alpha) = \Omega_6(\alpha)^{-1/6} k_f \Gamma^{1/3} \sqrt{g' V^{1/3}} \tag{13}$$

as stated in Webber et al. [1], but for a fixed cloud volume this becomes a function of α. The ratio between the velocity in the valley and the uniform slope velocity $(\Omega(0)/\Omega(\alpha))^{1/6}$ is shown in Fig. 4. The velocity increase with increasing α is due to the higher front when the lateral spreading is limited by the valley topography. The cloud advection velocity u is called the 'free-fall velocity' in Webber et al. [1], but perhaps 'terminal velocity' is a better designation since the front velocity $\sqrt{g'(h-a)}$ implicitly assumes a drag force.

3. Conclusion

Webber et al. [1] found an equilibrium state for the motion of a heavy gas cloud on a uniform slope. This result may be generalized to other terrain types, e.g. a v-shaped valley. The existence of an equilibrium state requires a constant slope in the x_1-direction but probably the lateral variation of the terrain height need just be symmetrical and concave. In addition to the solution in the v-shaped valley a numerical solution for a valley with a parabolic cross section is given in the Appendix. The SRD model does not include entrainment or surface friction, and the comparison with wind-tunnel data in Webber et al. [1] showed that the predicted velocity became too fast for steep slopes. It is still not known how sensitive the shape of the cloud is to surface friction, entrainment or ambient wind, but an immediate advantage of the SRD model is that it offers a test case for numerical shallow water models when the entrainment is switched off. Britter [2] suggests a model for steady continuous dense gas flow with friction and entrainment in an inclined valley.

4. Nomenclature

$a(\boldsymbol{x})$	height of the terrain
g'	reduced gravity
h	height of the top interface of the cloud
k_f	Froude number for the front velocity
L	x_1-coordinate of the front position
$\hat{L}$	$= L/\Lambda$
$\hat{\boldsymbol{n}}$	vector describing the terrain
u	cloud free-fall velocity
u_f	front velocity
V	cloud volume
$\boldsymbol{x}$	two-dimensional (x_1, x_2) position vector

y	x_2-coordinate of the front position
$\hat{y}$	$= y/\Lambda$
α	ratio of the terrain slope in the x_2 and x_1 direction
β	terrain parameter for a valley with a parabolic cross section
Γ	slope in the down-valley direction
Λ	cloud length scale
ω	a convenient parameter for determination of $\hat{L}$, $\hat{y}$
Ω_6	nondimensional cloud volume defined from $V = \Gamma\Lambda^3\Omega_6$

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Appendix. Motion in a valley with a parabolic cross section

A valley with a parabolic cross section is considered with the height of the terrain described by

$$a_*(x) = -\Gamma\left(x_1 - \frac{\beta}{\Lambda}x_2^2\right). \tag{A.1}$$

Here, the length scale of the cloud Λ has been used to formulate a problem with a nondimensional terrain parameter β, though in practice it is the ratio $\beta' = \beta/\Lambda$ which will be the known parameter. Following the same procedure as for the v-shaped valley above, the front coordinates ($\hat{L}_*$, $\hat{y}_*$) and the normalized cloud volume Ω_6^* are described by the system:

$$\hat{L}_* = \cos^2\omega + \beta\hat{y}_*^2,$$

$$\frac{\mathrm{d}\hat{y}_*}{\mathrm{d}\omega} = 2\cos^2\omega - 4\beta\frac{\hat{y}_*\cos^2\omega}{\tan\omega + 2\beta\hat{y}_*},$$

$$\Omega_6^*(\beta) = 2\int_0^{\pi/2} \cos^6\omega - 2\beta\frac{\hat{y}_*\cos^6\omega}{\tan\omega + 2\beta\hat{y}_*}\,\mathrm{d}\omega. \tag{A.2}$$

Presumably, $\hat{y}_*$ will not be a simple function of ω and analytical integration of Ω_6^* is probably impossible. However, the numerical integration of $\hat{y}_*$ and Ω_6^* is straightforward. An example of the resulting cloud shape is shown in Fig. 5 and cloud boundaries for different values of β are shown in Fig. 6. The normalized cloud volume and the cloud velocity are shown in Figs. 7 and 8, respectively. When the function $\Omega_6^*(\beta)$ is

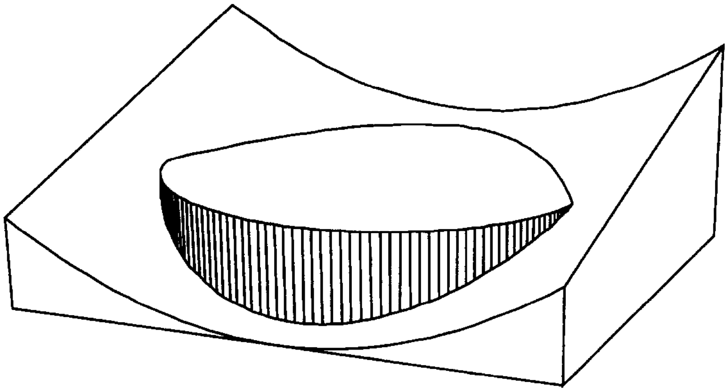

Fig. 5. Perspective of the model cloud in a valley with a parabolic cross section.

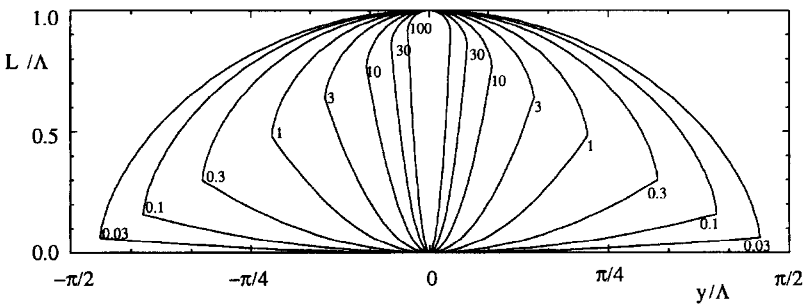

Fig. 6. Top view of cloud shapes in a parabolic valleys as a function of β.

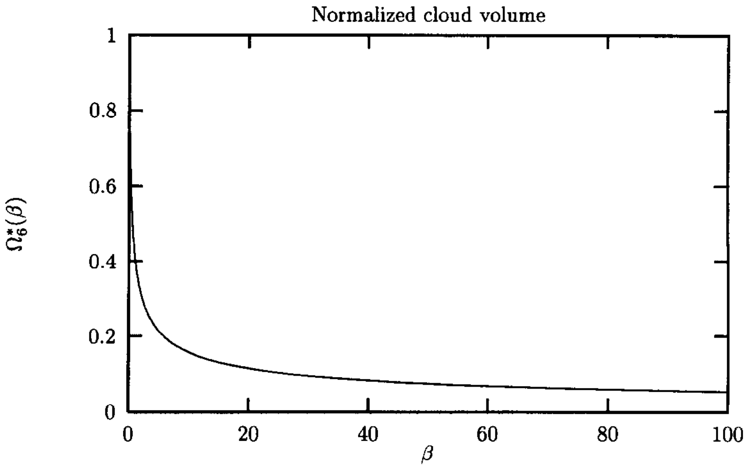

Fig. 7. The normalized cloud volume $\Omega_6(\beta)$ in a valley with a parabolic cross section.

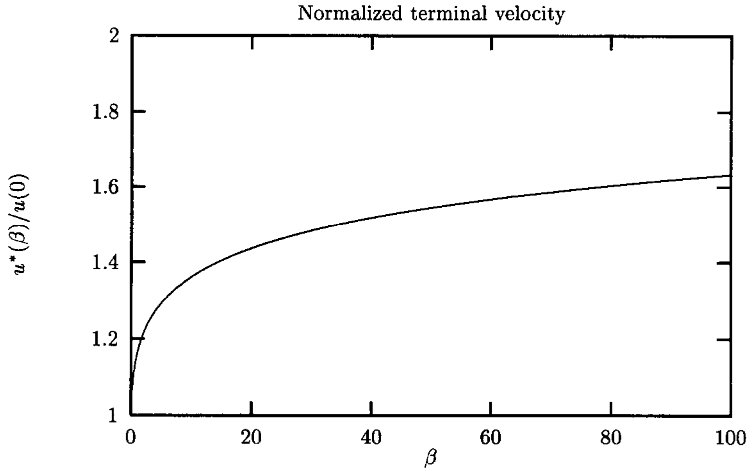

Fig. 8. The normalized velocity of the cloud $u(\beta)/u(0)$ in a valley with a parabolic cross section.

known, the proper value of β must be found from the equation

$$\beta = \beta' \left(\frac{V}{\Gamma \Omega_6^*(\beta)} \right)^{1/3}. \tag{A.3}$$

References

[1] D.M. Webber, S.J. Jones and D. Martin, A model of the motion of a heavy gas cloud released on a uniform slope, J. Hazard. Mater., 33 (1993) 101–122.

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